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Exponential equations

Revision:

Properties of Exponents

For any real numbers a and b and any integers m and n, assuming 0 is not raised to a nonpositive power:

$a^m \cdot a^n = a^{m+n}$	Product rule
$\frac{a^m}{a^n} = a^{m-n}$ $(a \neq 0)$	Quotient rule
$(a^m)^n = a^{mn}$	Power rule
$(ab)^m = a^m b^m$	Raising a product to a power
$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ $(b \neq 0)$	Raising a quotient to a power

Rational Exponents

For any real number a and any natural numbers m and n, $n \geq 2$, for which $\sqrt[n]{a}$ exists,

$$a^{1/n} = \sqrt[n]{a},$$

$$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m, \quad \text{and}$$

$$a^{-m/n} = \frac{1}{a^{m/n}}.$$

Solving Exponential Equations

Equations with variables in the exponents, such as

$$3^x = 20 \quad \text{and} \quad 2^{5x} = 64,$$

are called **exponential equations.**

Sometimes, as is the case with the equation $2^{5x} = 64$, we can write each side as a power of the same number:

$$2^{5x} = 2^6.$$

$$5x = 6$$

$$x = \frac{6}{5}$$

LS: $2^{5\frac{6}{5}} = 2^6 = 64$ RS: 64

LS = RS

We use the following property.

Base–Exponent Property

For any $a > 0$, $a \neq 1$,

$$a^x = a^y \longleftrightarrow x = y.$$

This property follows from the fact that for any $a > 0$, $a \neq 1$, $f(x) = a^x$ is a one-to-one function. If $a^x = a^y$, then $f(x) = f(y)$. Then since f is one-to-one, it follows that $x = y$. Conversely, if $x = y$, it follows that $a^x = a^y$, since we are raising a to the same power.

Exercises

Solve $5^x = 5^3$

Since the bases ("5" in each case) are the same, then the only way the two expressions could be equal is for the powers also to be the same. That is:

$$\mathbf{x = 3}$$

Solve $10^{1-x} = 10^4$

Since the bases are the same, I can equate the powers and solve:

$$1 - x = 4$$
$$1 - 4 = x$$
$$\mathbf{-3 = x}$$

Solve $3^x = 9$

Since $9 = 3^2$, this is really asking me to solve: $3^x = 3^2$

Since the bases are the same, I can set the powers equal:

$$\mathbf{x = 2}$$

Solve $3^{2x-1} = 27$.

In this case, I have an exponential on one side of the "equals" and a number on the other. I can solve this if I can express "27" as a power of 3. Since $27 = 3^3$, then I can proceed:

$$3^{2x-1} = 27$$
$$3^{2x-1} = 3^3$$
$$2x - 1 = 3$$
$$2x = 4$$
$$\mathbf{x = 2}$$

As you can probably tell, you will need to get good with your powers of numbers, such as the powers of 2 up through $2^6 = 64$, the powers of 3 up through $3^5 = 243$, the powers of 4 up through $4^4 = 256$, the powers of 5 up through $5^4 = 625$, the powers of 6 up through $6^3 = 216$, and all the squares. Don't plan to depend on your calculator; you'll want to have a certain degree of facility on the test, so familiarize yourself with the smaller powers now.

Solve $3^{x2-3x} = 81$.

This one works just like the previous example:

$$3^{x^2-3x} = 81$$
$$3^{x^2-3x} = 3^4$$
$$x^2 - 3x = 4$$
$$x^2 - 3x - 4 = 0$$
$$(x - 4)(x + 1) = 0$$
$$\boldsymbol{x = -1 \text{ and } 4}$$

Solve $4^{2x^2+2x} = 8$.

This one is similar to the previous two, but not quite the same, because 8 is not a power of 4. However, both 8 and 4 are powers of 2, so convert:

$$4 = 2^2$$
$$8 = 2^3$$
$$4^{2x^2+2x} = \left(2^2\right)^{2x^2+2x} = 2^{4x^2+4x}$$

Now I can solve:

$$4^{2x^2+2x} = 8$$
$$2^{4x^2+4x} = 2^3$$
$$4x^2 + 4x = 3$$
$$4x^2 + 4x - 3 = 0$$
$$(2x - 1)(2x + 3) = 0$$
$$\boldsymbol{x = \frac{1}{2} \text{ and } -\frac{3}{2}}$$

Solve $4^{x+1} = {}^1/_{64}$.

Recall that negative exponents mean "flip the base to the other side of the fraction line". Recall also that $64 = 4^3$. Then ${}^1/_{64} = (4^3)^{-1} = 4^{-3}$. With that, I can solve:

$$4^{x+1} = \frac{1}{64}$$
$$4^{x+1} = 4^{-3}$$

$x + 1 = -3$
$\mathbf{x = -4}$

- **Solve $8^{x-2} = \sqrt{8}$**

  Recall that square roots are one-half powers, and convert:

  $$8^{x-2} = \sqrt{8}$$
  $$8^{x-2} = 8^{\frac{1}{2}}$$
  $$x - 2 = \frac{1}{2}$$
  $$\mathbf{x = \frac{5}{2}}$$

By the way, the following is a common type of trick question:

Solve $2^x = -4$

Think about it: What power on the *positive* number "2" could *possibly* yield a *negative* number? I'll never go from positive to negative by taking powers; I can never turn a positive two into a negative anything, four or otherwise, by multiplying two by itself, regardless of the number of times I do the multiplication. Exponentiation just doesn't work that way. So the answer here is:

no solution