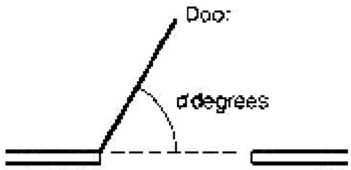
Derivative, Geometrical meaning of derivative, Differentiation

YEAR5 MAT/ECO

The Concept of Instantaneous Rate

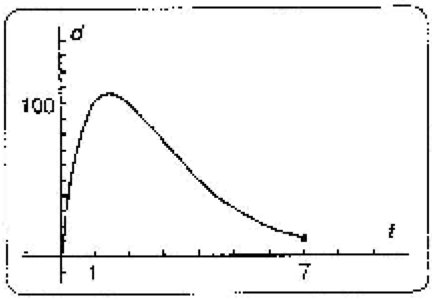
If you push open a door that has an automatic closer, it opens fast at first, slows down, stops, starts closing, then slams shut. As the door moves, the number of degrees it is from its closed position depends on how many seconds it has been since you pushed it.



The questions to be answered are, “At any particular instant in time, is the door opening or closing?” and, ‘How fast is it moving?”

Given the equation for a function relating two variables, estimate **the instantaneous rate of change** of the dependent variable with respect to the independent variable **at a given point**. Suppose that a door is pushed open at time t = 0 seconds, and slams shut again at time t = ‘*7* seconds. While the door is in motion, assume that the number of degrees, ***d***, from its closed position is modeled by the following equation:

$d = 200t \cdot 2^{-t}$, for $0 \le t \le 7$



How fast is the door moving at the instant when t = 1 second? When t is 1, the graph is going up as t increases from left to right. So the angle is getting bigger and the door is opening. You can estimate the rate numerically by calculating values of *d* for values of t close to 1.

$t = 1:\ d = 200(1)\cdot 2^{-1} = 100$ degrees

$t = 1.1: d = 200(1.1)\cdot 2^{-1.1} = 102.633629$ degrees

So the door’s angle increased by 2.633 degrees in 0.1 second, meaning that it moved at a rate of about (2.633...)/0,1, or 26.33 degrees per second. But this rate is an *average* rate, and the question was about **an instantaneous rate**. In an “instant,” 0 seconds long, the door moves 0degrees. So the rate would be 0/0, which is awkward because of division by zero.
To get closer to the instantaneous rate at t = 1, find d at t = 1.01 and at t = 1.001 seconds.

t = 1,01: $d = 200(1.01).2^{-1,01} = 100.30234$, a change of 0.30234 degrees

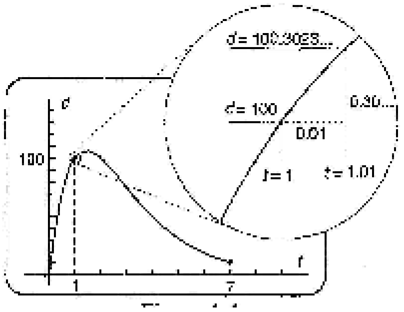
t = 1,001: $d = 200(1.001).2^{-1,001} = 100.03064$, a change of 0.03064 degrees

The average rates for the time intervals 1 to 1.01 and 1 to 1.001 seconds are

1 to 1.01 seconds: Average rate = $\dfrac{100{,}30234 - 100}{0{,}01}$ = 30.234 degrees/second

1 to 1.001 seconds: Average rate = $\dfrac{100{,}03064 - 100}{0{,}001}$ = 30.64 degrees/second

The important thing for you to notice is that as the time interval gets smaller and smaller, the number of degrees per second doesn't change much.



As you zoom in on the point (1, 100), the graph appears to be straighter. So the change in d divided by the change in t becomes closer to the slope of a straight line.

If you list the average values in a table, another interesting feature appears. The values stay the same for more and more decimal places.

<u>Seconds</u>	<u>Average rate</u>
1 to 1.01	30.23420...
1 to 1.001	30.64000...
1 to 1.0001	*30.68075...*
1 to 1.00001	30.68482...
1 to 1.000001	30.68524...

There seems to be ***a limiting number*** that the values are approaching.
Estimating the instantaneous rate at $t = 3$ gives the following results.

t=3:	$d = 200(3).2^{-3}$	= 75 degrees.
t=3.1:	$d = 200(3.1).2^{-3,1}$	= 72.3 10056 degrees.
t=3.01:	$d = 200(3.01).2^{-3,01}$	= 74.730210 degrees.
t=3.001:	$d = 200(3.001).2^{-3,001}$	= 74.973014 degrees.

The corresponding average rates are
3 to 3.1 seconds: Average rate = -26.899 degrees/second
3 to 3.01 seconds: Average rate = -26.978 degrees/second
3 to 3.001 seconds: Average rate = -26.985 degrees/second
Again the rates seem to be getting closer to some limiting number, this time around -27. So the instantaneous rate at t = 3 seconds should be somewhere close to -27 degrees/second. The negative sign tells you that the number of degrees, d, is decreasing as time goes on. Thus the door is closing when t = 3. It is opening when t = 1 because the rate of change is positive.

For the door example, above, the angle is said to be a function of time. **Time is the independent variable and angle is dependent variable.** The names make sense because the number of degrees the door is open depends on the number of seconds since it was pushed. **The instantaneous rate of change of the dependent variable is said to be the limit of the average rates as the time interval gets closer to zero. This limiting value is called the derivative of the dependent variable with respect to the independent one.**

<u>Meaning of Derivative</u>

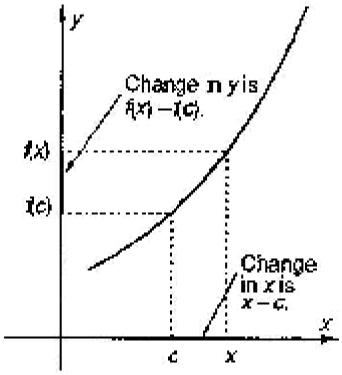
The derivative of a function at a particular value of the independent variable is the ***instantaneous rate of change*** of the dependent variable with respect to the independent variable.

Derivative, Geometrical meaning of derivative, Differentiation

YEAR5 MAT/ECO

Difference Quotients and Definition of Derivative

You have been finding derivatives by taking a change in x, dividing it into the corresponding change in y, and taking the limit of the resulting fraction as the change in x approaches zero.



The change in the y-value is equal to *f(x)* —*f(c)*. The change in the x-value is x — *c*.

So the derivative is approximately equal to $\frac{f(x)-f(c)}{x-c}$

this fraction is called **a difference quotient.**

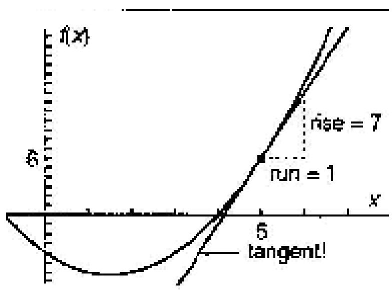
The derivative is the function you get by taking the limit of the difference quotient as the denominator approaches zero.
If the function's name is f, then the symbol f ` (pronounced '*f* prime") is often used for the derivative function. The symbol shows that there is a relationship, yet a difference between the original function and the function "derived" from it (hence the name "derivative") which tell its rate of change.

Definition of Derivative (derivative at x = *c* form)

$f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ Meaning: The instantaneous rate of change of f(x)) with respect to x at x = *c*.

Geometrical Interpretation of Derivative: Slope of a Tangent Line

The derivative of a function at a point equals the slope of the tangent line to the graph of the function at that point. Both equal the instantaneous rate of change.



The process of finding an equation for the derivative of a function is called **differentiation**. The word is in honor of the fact that $\Delta y/\Delta x$ is a **difference quotient**. The corresponding verb is "**differentiate"**.

There are other widely-used symbols for the derivative, with which you should become familiar. The symbol **dy/dx** comes from $\Delta y/\Delta x$ the difference quotient, and means that the **limit** is to be taken as Δy and Δx both go to zero. It is the terminology used by Leibniz starting in about *1675.* Later *dy* and *dx* are called "differentials." For the time being, *dyldx* will be regarded as a *single* **symbol that cannot be taken apart**. Avoid saying *"dy* over *dx,"* or people may think you are rather naive!

The symbol $\frac{d}{dx}$ comes from rearranging the letters in $\frac{dy}{dx}$. It is an operator that acts on the expression y, similar to sin y or log y. It tells you to take the derivative of y with respect to x.

$\frac{dy}{dx}$, pronounced "dee y, dee x." (A *single* symbol, not a fraction!)

$\frac{d}{dx}$(y), pronounced "dee, dee x, of y" (An *operation* done on y)