

Analytical geometry of linear shapes

YEAR 3

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Basic terms

A *line* is given by 2 different points or one point and a vector.

The *line segment* (*abscissa*, pl. *abscissae*) is the straight line joining some point A to some other point B.

A *ray* is a line with a start point but no end point (it goes to infinity).

A *plane* is a two-dimensional surface. It is given by three noncollinear (do not lie on the same line) points or by a point and two noncollinear vectors.

Space is a three-dimensional or **3D** because there are three dimensions: *width*, *depth* and *height*.

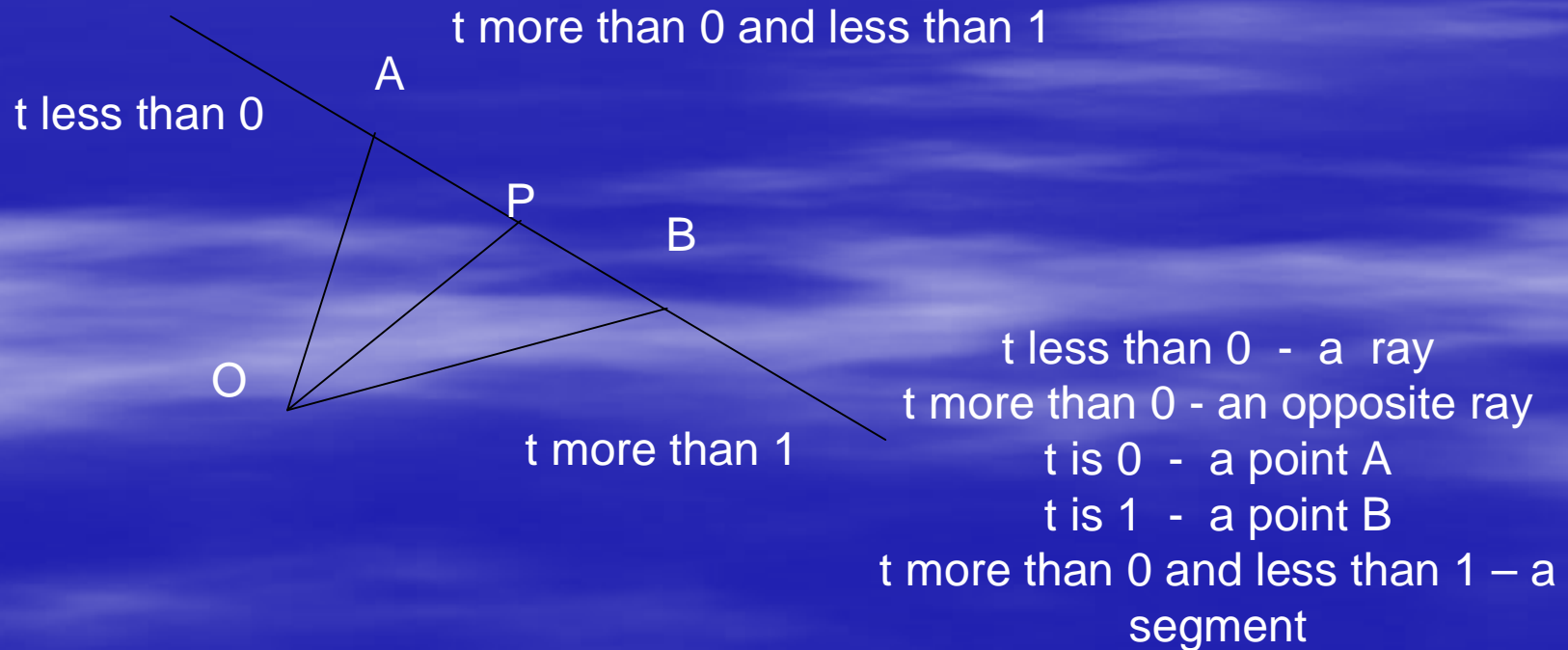
A direction vector shows the direction of a line.

A normal vector is a perpendicular vector to the direction of the line.

Vector equation of a line

Let OA and OB be the position vectors of two points A and B with respect to an origin O . Let OP be the position vector of a point P on the line AB .

$OP = OA + AP = OA + t AB$, where t is a scalar (parameter).



Vector equation of a line in a plane

The line is t multiple of the line AB

$$AP = t (B - A)$$

$$P - A = t (B - A) \quad /+A$$

$$P = A + t (B - A)$$

Any point of the line can be found as one point of the line plus t times direction vector.

$$[x, y] = [x_A, y_A] + t (x_B - x_A; y_B - y_A)$$

is a vector equation of a line AB.

The parametric equations of a line in a plane

Given two points $A=[x_1, y_1]$ and $B=[x_2, y_2]$
and vector $AB=(x_2-x_1, y_2-y_1)$,

$$P = A + t AB$$

$$[x, y] = [x_A, y_A] + t (x_B - x_A; y_B - y_A)$$

(Distinguish x and y coordinates)

The parametric equations of a line in a plane are:

$$x = x_1 + t (x_2 - x_1),$$

$$y = y_1 + t (y_2 - y_1), \text{ } t \text{ belongs to } \mathbb{R}$$

The midpoint formula

Let $A=[x_1, y_1]$ and $B=[x_2, y_2]$ be two points. The midpoint between them has coordinates

$$S = [(x_A+x_B)/2; (y_A+y_B)/2]$$

Exercise 1

Find equations of the line that passes through $A = [-2,1]$ and $B = [1,3]$ and find the coordinates of the midpoint.

(a) Direction Vector: $B - A = (3,2)$.

(b) Vector Equation:

$$[x, y] = [-2, 1] + t(3, 2)$$

(c) Parametric Equations:

$$\begin{aligned}x &= -2 + 3t, \\y &= 1 + 2t, \quad t \text{ belongs to } \mathbb{R}\end{aligned}$$

(d) Midpoint: $[(1-2)/2; (1+3)/2]$
 $[-0,5; 2]$

Exercise 2

Find out if a point $C=[1;2]$ lies on the line that passes through $A = [-2,1]$ and $B= [1,3]$.

Parametric Equations are $x = -2 + 3t$,
 $y = 1 + 2t$, t belongs to \mathbb{R}

Put coordinates of C instead of x and y .

$$1 = -2 + 3t, \text{ so } t_1 = 1$$

$$2 = 1 + 2t, \text{ so } t_2 = 0,5$$

t_1 is different from t_2 so C doesn't lie on this line.

Vector equation of a line in space

The line is t multiple of the line AB

$$\mathbf{AP} = t (\mathbf{B} - \mathbf{A})$$

$$\mathbf{P} - \mathbf{A} = t (\mathbf{B} - \mathbf{A}) \quad / +\mathbf{A}$$

$$\mathbf{P} = \mathbf{A} + t (\mathbf{B} - \mathbf{A})$$

Any point of the line can be found as one point of the line plus t times direction vector.

$$[x, y, z] = [x_A, y_A, z_A] + t (x_B - x_A; y_B - y_A; z_B - z_A)$$

is a vector equation of a line AB in space.

The parametric equations of a line in space

Given two points $A=[x_1, y_1, z_1]$ and $B=[x_2, y_2, z_2]$
and vector $AB=(x_2-x_1, y_2-y_1, z_2-z_1)$,

$$P = A + t \mathbf{AB}$$

$$[x, y, z] = [x_A, y_A, z_A] + t (x_B - x_A; y_B - y_A; z_B - z_A)$$

(Distinguish x and y coordinates)

The parametric equations of a line in a plane are:

$$x = x_1 + t (x_2 - x_1),$$

$$y = y_1 + t (y_2 - y_1),$$

$$z = z_1 + t (z_2 - z_1), \text{ t belongs to } \mathbb{R}$$

The Cartesian equation of a line (general form)

$$a x + b y + c = 0,$$

where a , b , and c are real numbers, and both a and b are not zero

a and b are components of a normal vector of a line

x and y are coordinates of a point

c is a y -axis intercept

The Cartesian equation of a line (general form)

The Cartesian equation of the line can be found from the parametric form.

The vector equation of the line is

$$[x, y] = [3, 1] + t(1, -2)$$

The parametric equations of the line are

$$x = 3 + t,$$

$$y = 1 - 2t, \text{ } t \text{ belongs to } \mathbb{R}$$

Eliminate t term (use addition or substitution method).

$$x = 3 + t, \text{ so } t = x - 3$$

$$y = 1 - 2t,$$

$$y = 1 - 2(x - 3)$$

$$y = 1 - 2x + 6$$

Put everything to one side.

$$y + 2x - 7 = 0$$

The Cartesian equation of the line is **$2x + y - 7 = 0$**

The Cartesian equation of a line (general form) in a plane

The other way how to find the Cartesian equation of the line is to find a normal vector \mathbf{n} (a normal vector of the line) that is perpendicular to vector

$$\mathbf{v} (1,-2) \quad \mathbf{n} \cdot \mathbf{v} = 0$$

if $\mathbf{a} = (a_1, a_2)$ then \mathbf{n} is $(a_2, -a_1)$

change x and y component, then change the sign before one component

$$\mathbf{v} = (1,-2) \text{ then } \mathbf{n} = (-2, -1) \text{ or } (2, 1)$$

point A lies on the line so

$$-2(3) - 1(1) + c = 0$$

$$-6 - 1 = -c$$

$$c = 7$$

The Cartesian equation of the line is $-2x - y + 7 = 0$

It is -1 multiple of $2x + y - 6 = 0$

They express the same line.

The Cartesian equation of a line (general form) in space

There is no Cartesian equation of a line in space.

It can be written just for the line in a plane.